Numerical approximation of the free-congested Navier-Stokes equations

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Free - congested Navier-Stokes equations

compressible fluid under maximal density constraint in a bounded domain $\Omega \subset \mathbb{R}^3$

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p - \mu \Delta u - (\lambda + \mu) \nabla \mathrm{div}\, u = 0 \\ 0 \le \rho \le 1, \ (1-\rho)p = 0, \ p \ge 0 \end{cases}$$

$$+ \text{ Dirichlet BC: } \quad u_{|\partial\Omega} = 0$$

- $\rho < 1$: free / pressureless equations
- $\rho = 1$: constrained / incompressible equations

  p is the Lagrange multiplier associated to constraint $\mathrm{div}\, u = 0$
- the interface between the free domain and congested domain is unknown

Motivation

- asymptotic model for two-phase flows when the ratio $\frac{\rho_1}{\rho_2} \to 0$

  Bouchut, Brenier, Cortes, Ripoll (2001)

- partially pressurized flows / floating structures

  Bourdarias, Ersoy, Gerbi (2012)
  Lannes, Bocchi (2017, 2019)
  Godlewski, Parisot, Sainte-Marie, Wahl (2018, 2019)

- collective motion

  Degond, Hua, Navoret (2011), Maury et al. (2012) → see also Hele-Shaw models

several approximations proposed to approach the free boundary problem

Approximations - inviscid case

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p = 0 \\ 0 \le \rho \le 1, \ (1-\rho)p = 0, \ p \ge 0 \end{cases}$$

- "sticky" blocks

  ref: Berthelin (2002), Preux (2016)

- relaxation/penalty approach (convergence not proved)

$$p_\lambda(\rho) = \frac{(\rho - 1)_+}{\lambda^2}, \qquad \lambda \to 0$$

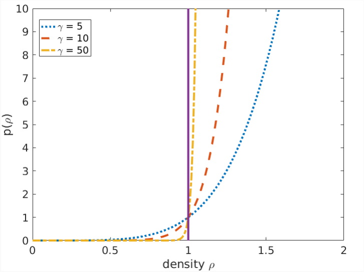
  ref: Godlewski, Parisot, Sainte-Marie, Wahl (2018-2019)

Approximations - viscous case

penalty approach

- penalization of large densities

$$p_n(\rho) = a\rho^{\gamma_n}, \ \gamma_n \to +\infty$$

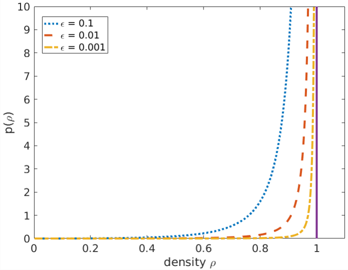


ref: Lions, Masmoudi '99

soft congestion approach

- singular pressure → barrier

$$p_\varepsilon(\rho) = \varepsilon \frac{\phi(\rho)}{(1-\rho)^\beta}, \ \varepsilon \to 0$$



ref: Degond et al. '11, P. & Zatorska '15

convergence of weak solutions as $n \to +\infty$ / $\varepsilon \to 0$

- uniform estimates: energy, control of the pressure
- convergence via the effective flux and the use of renormalized solutions

Soft approximation of the free boundary problem

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$
$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p - \mu \Delta u - (\lambda + \mu)\nabla \operatorname{div} u = 0$$

$$0 \le \rho \le 1, \quad (1-\rho)p = 0, \quad p \ge 0$$

soft approx.: compressible NS eq. with singular pressure $p_\varepsilon(\rho) = \varepsilon \frac{\rho^\gamma}{(1-\rho)^\beta}$, $\gamma, \beta > 1$

$$(\mathscr{P}_\varepsilon) \qquad \begin{aligned} &\partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0 \\ &\partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla p_\varepsilon(\rho_\varepsilon) - \mu \Delta u_\varepsilon - (\lambda + \mu)\nabla \operatorname{div} u_\varepsilon = 0 \\ &0 \le \rho_\varepsilon < 1 \quad \text{a.e.} \end{aligned}$$

main goal: discretization of $(\mathscr{P}_\varepsilon)$ which "preserves" the asymptotics $\varepsilon \to 0$

Continuous problem

$$(\mathscr{P}_\varepsilon) \qquad \begin{aligned} &\partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0 \\ &\partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla p_\varepsilon(\rho_\varepsilon) - \mu \Delta u_\varepsilon - (\lambda+\mu)\nabla \operatorname{div} u_\varepsilon = 0 \\ &0 \le \rho_\varepsilon < 1 \ \text{ a.e.}, \quad p_\varepsilon(\rho) = \varepsilon \frac{\rho^\gamma}{(1-\rho)^\beta} \end{aligned}$$

- construction of $(\rho_\varepsilon, u_\varepsilon)$ → truncation of the pressure law $p_{\varepsilon,\delta}$, $\delta \to 0$

$$p_{\varepsilon,\delta}(\rho) = \begin{cases} p_\varepsilon(\rho) & \text{if } \rho \le 1-\delta \\ \varepsilon \delta^{-\beta} \rho^\gamma & \text{if } \rho > 1-\delta \end{cases}$$

- uniform estimates on $(\rho_\varepsilon, u_\varepsilon)$
  - renormalized mass eq. with $H_\varepsilon'(\rho)\rho - H_\varepsilon(\rho) = p_\varepsilon(\rho)$, $H_\varepsilon(\rho) \underset{1^-}{\sim} \varepsilon(1-\rho)^{1-\beta}$

$$\partial_t H_\varepsilon(\rho_\varepsilon) + \operatorname{div}\big(H_\varepsilon(\rho_\varepsilon) u_\varepsilon\big) = -p_\varepsilon(\rho_\varepsilon) \operatorname{div} u_\varepsilon$$

  - energy estimate

$$\sup_{t \in [0,T]} \int_\Omega \left[\frac{\rho_\varepsilon |u_\varepsilon|^2}{2} + H_\varepsilon(\rho_\varepsilon) \right] + \int_0^T \int_\Omega \left[\mu\, |\nabla u_\varepsilon|^2 + (\lambda+\mu)(\operatorname{div} u_\varepsilon)^2 \right] \le E_0$$

Continuous problem - Control of the pressure

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0 \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla p_\varepsilon(\rho_\varepsilon) - \mu\Delta u_\varepsilon - (\lambda+\mu)\nabla \operatorname{div} u_\varepsilon = 0 \\ 0 \le \rho_\varepsilon < 1 \end{cases}$$

- use as test function in the mom. eq.: $\psi = \mathscr{B}(\rho_\varepsilon - <\rho_\varepsilon>)$, $\mathscr{B} \approx \operatorname{div}^{-1}$

$$\left| \int_0^T \int_\Omega p_\varepsilon(\rho_\varepsilon)(\rho_\varepsilon - <\rho_\varepsilon>) \right| \le C(E^0)$$

- split the integral into two parts: $\{\rho_\varepsilon \le \frac{1+<\rho_\varepsilon>}{2}\}$ and $\{\rho_\varepsilon > \frac{1+<\rho_\varepsilon>}{2}\}$

if $<\rho_\varepsilon> = <\rho_\varepsilon^0> \le \rho^* < 1$ then

$$\begin{aligned} C &\ge \int_0^T \int_\Omega p_\varepsilon(\rho_\varepsilon)(\rho_\varepsilon - <\rho_\varepsilon>) 1_{\{\rho_\varepsilon > \frac{1+<\rho_\varepsilon>}{2}\}} \\ &\ge \frac{1-<\rho_\varepsilon>}{2} \int_0^T \int_\Omega p_\varepsilon(\rho_\varepsilon) 1_{\{\rho_\varepsilon > \frac{1+<\rho_\varepsilon>}{2}\}} \\ &\ge \frac{1-\rho^*}{2} \int_0^T \int_\Omega p_\varepsilon(\rho_\varepsilon) 1_{\{\rho_\varepsilon > \frac{1+<\rho_\varepsilon>}{2}\}} \end{aligned}$$

$$\Longrightarrow \boxed{\|p_\varepsilon(\rho_\varepsilon)\|_{L^1([0,T]\times\Omega)} \le C}$$

Continuous problem - Limit

- convergences

$$
\begin{aligned}
\rho_\varepsilon \to \rho \quad & \text{strongly in } L^q\big((0,T)\times\Omega\big)\ \forall\ 1\le q<+\infty,\\
u_\varepsilon \to u \quad & \text{weakly in } L^2(0,T;H^1_0(\Omega)),\\
p_\varepsilon(\rho_\varepsilon)\to p \quad & \text{weakly in } \mathscr{M}_+\big((0,T)\times\Omega\big)
\end{aligned}
$$

- exclusion relation

$$
(1-\rho_\varepsilon)p_\varepsilon(\rho_\varepsilon)\to 0 \quad \text{strongly in } L^q((0,T)\times\Omega) \text{ for some } q>1
$$

- (ρ,u,p) is a weak solution to

$$
\begin{cases}
\partial_t\rho + \mathrm{div}(\rho u) = 0\\
\partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) + \nabla p - \mu\Delta u - (\lambda+\mu)\nabla\,\mathrm{div}\, u = 0\\
0\le\rho\le 1, \quad (1-\rho)p = 0, \quad p\ge 0
\end{cases}
$$

ref: P., Zatorska (2015)

Numerical approximation of the model

$$
(\mathscr{P}_\varepsilon) \qquad \begin{cases} \partial_t \rho_\varepsilon + \mathrm{div}(\rho_\varepsilon u_\varepsilon) = 0 \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \mathrm{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla p_\varepsilon(\rho_\varepsilon) - \mu\Delta u_\varepsilon - (\lambda+\mu)\nabla \mathrm{div}\, u_\varepsilon = 0 \\ 0 \le \rho_\varepsilon < 1 \ \text{ a.e.}, \quad p_\varepsilon(\rho) = \varepsilon \dfrac{\rho^\gamma}{(1-\rho)^\beta} \end{cases}
$$

- in the regions where $1-\rho_\varepsilon \ll 1$ ≈ incompressible regime,

$$\text{characteristic speeds } = u \pm \sqrt{p'_\varepsilon(\rho_\varepsilon)}$$

  - if the scheme is explicit ⇝ instable scheme because the CFL is too stringent
  - colocated schemes loose precision in the dense regions (high num. diffusion) possible corrections → see Degond, Hua, Navoret (2011)

Numerical approximation of the model

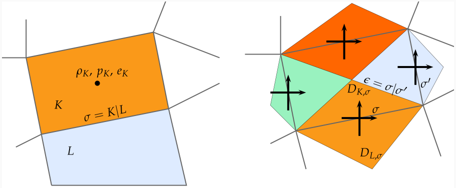
$$(\mathscr{P}_\varepsilon) \qquad \begin{aligned} &\partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0 \\ &\partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla p_\varepsilon(\rho_\varepsilon) - \mu \Delta u_\varepsilon - (\lambda + \mu) \nabla \operatorname{div} u_\varepsilon = 0 \\ &0 \le \rho_\varepsilon < 1 \ \text{ a.e.,} \quad p_\varepsilon(\rho) = \varepsilon \frac{\rho^\gamma}{(1-\rho)^\beta} \end{aligned}$$

- staggered FV/FE space discr. + projection method (open software CALIF^{3}S)
    - hybrid method Finite Volume - Finite Elemente
    - staggered scheme $\Longrightarrow$ discrete Bogovskii operator (inf-sup condition)
      $\Longrightarrow$ stability of the pressure
    - pressure correction scheme adapted to compressible equations

  analogy with the low Mach number limit: Herbin, Latché, Saleh (2020)

Implicit staggered FV-FE scheme

- $\Omega \subset \mathbb{R}^3$ polyhedral, $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ a staggered discretization



- scalars at the centers of cells: $(\rho_K)_{K \in \mathcal{M}}$
- vectors at the interfaces: $(u_\sigma)_{\sigma \in \mathcal{E}}$

- mass eq.: FV on the primal mesh

$$\boxed{\frac{1}{\delta t}(\rho_K^{n+1} - \rho_K^n) + \mathrm{div}(\rho^{n+1} u^{n+1})_K = 0}$$

- mom eq.: mixed FV - FE (Crouzeix-Raviart / Rannacher-Turek) on the dual mesh

$$\boxed{\frac{1}{\delta t}(\rho_{D_\sigma}^{n+1} u_\sigma^{n+1} - \rho_{D_\sigma}^n u_\sigma^n) + \mathrm{div}(\rho^{n+1} u^{n+1} \otimes u^{n+1})_\sigma - \mathrm{div}(\tau(u^{n+1}))_\sigma + (\nabla p_{\varepsilon,h}^{n+1})_\sigma = 0}$$

where $|D_\sigma|\rho_{D_\sigma} = |D_{K,\sigma}|\rho_K + |D_{L,\sigma}|\rho_L \quad \forall \sigma \in \mathcal{E}_{\mathrm{int}},\ \sigma = K|L$

$$p_{\varepsilon,h}(\rho) = \varepsilon \delta_{\varepsilon,h}^{-\beta} \rho^\gamma \quad \text{for } \rho > 1 - \delta_{\varepsilon,h}$$

Discrete renormalized mass equation

- continuous case:

$$\partial_t H_\varepsilon(\rho_\varepsilon) + \mathrm{div}(H_\varepsilon(\rho_\varepsilon) u_\varepsilon) = -p_\varepsilon(\rho_\varepsilon)\,\mathrm{div}\, u_\varepsilon \quad \text{with} \quad H'_\varepsilon(\rho)\rho - H_\varepsilon(\rho) = p_\varepsilon(\rho)$$

- discrete mass equation

$$\frac{1}{\delta t}(\rho_K^{n+1} - \rho_K^n) + \mathrm{div}(\rho^{n+1} u^{n+1})_K = 0$$

$$\text{with} \quad \mathrm{div}(\rho u)_K = \frac{1}{|K|} \sum_{\sigma \in \mathscr{E}(K)} F_{K,\sigma}, \quad F_{K,\sigma} = |\sigma| \rho_\sigma u_\sigma \cdot n_{K,\sigma}, \quad \rho_\sigma = \rho_\sigma^{\mathrm{up}}$$

  - define $H_{\varepsilon,h}$ such that $H'_{\varepsilon,h}(\rho)\rho - H_{\varepsilon,h}(\rho) = p_{\varepsilon,h}(\rho)$

$$\boxed{\frac{1}{\delta t}\Big(H_{\varepsilon,h}(\rho_K^{n+1}) - H_{\varepsilon,h}(\rho_K^n)\Big) + \mathrm{div}(H_{\varepsilon,h}(\rho^{n+1}) u^{n+1})_K + R_K^{n+1} = -p_{\varepsilon,h}(\rho_K^{n+1})\,\mathrm{div}(u^{n+1})_K}$$

$$H_{\varepsilon,h} \text{ convex} \implies \sum_{K \in \mathscr{M}} |K| R_K^{n+1} \geq 0$$

Discrete energy estimate - I

$$\frac{1}{\delta t}(\rho_K^{n+1} - \rho_K^n) + \mathrm{div}(\rho^{n+1} u^{n+1})_K = 0$$
$$\frac{1}{\delta t}(\rho_{D_\sigma}^{n+1} u_\sigma^{n+1} - \rho_{D_\sigma}^n u_\sigma^n) + \mathrm{div}(\rho^{n+1} u^{n+1} \otimes u^{n+1})_\sigma - \mathrm{div}(\tau(u^{n+1}))_\sigma + (\nabla p_{\varepsilon,h}^{n+1})_\sigma = 0$$

Goal: Multiply by u_σ^{n+1} the mom. eq., sum over $\sigma \in \mathcal{E}$ / combine with the mass eq.

- $\nabla - \mathrm{div}$ discrete duality:

$$\sum_{K \in \mathcal{M}} |K| p_K \,\mathrm{div}(u)_K = \sum_{K \in \mathcal{M}} |K| p_K \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| u_\sigma \cdot n_{K,\sigma}$$
$$= -\sum_{\sigma \in \mathcal{E}} |\sigma| (p_L - p_K) u_\sigma \cdot n_{K,\sigma}$$

Setting $(\nabla p)_\sigma = \dfrac{|\sigma|}{|D_\sigma|}(p_L - p_K) n_{K,\sigma}$ for $\sigma = K|L$, we ensure

$$\boxed{\sum_{K \in \mathcal{M}} |K| p_K \,\mathrm{div}(u)_K = -\sum_{\sigma \in \mathcal{E}} |D_\sigma| u_\sigma \cdot (\nabla p)_\sigma}$$

Discrete energy estimate - II

$$\frac{1}{\delta t}(\rho_K^{n+1} - \rho_K^n) + \frac{1}{|K|}\sum_{\sigma\in\mathscr{E}(K)} F_{K,\sigma}(\rho^{n+1}, u^{n+1}) = 0$$

$$\frac{1}{\delta t}(\rho_{D_\sigma}^{n+1} u_\sigma^{n+1} - \rho_{D_\sigma}^n u_\sigma^n) + \mathrm{div}(\rho^{n+1} u^{n+1} \otimes u^{n+1})_\sigma - \mathrm{div}(\tau(u^{n+1}))_\sigma + (\nabla p_{\varepsilon,h}^{n+1})_\sigma = 0$$

- we define $|D_\sigma|\,\rho_{D_\sigma} = |D_{K,\sigma}|\,\rho_K + |D_{L,\sigma}|\,\rho_L$

$$\mathrm{div}(\rho u \otimes u)_\sigma = \frac{1}{|D_\sigma|}\sum_{\epsilon\in\widetilde{\mathscr{E}}(D_\sigma)} F_{\sigma,\epsilon}\cdot u_\epsilon, \quad u_\epsilon = \frac{1}{2}(u_\sigma + u_{\sigma'}) \text{ for } \epsilon = D_\sigma|D_{\sigma'}$$

with $F_{K,\sigma} + \sum_{\epsilon\in\mathscr{E}(D_\sigma),\epsilon\subset K} F_{\sigma,\epsilon} = \frac{|D_{K\sigma}|}{|K|}\sum_{\sigma\in\mathscr{E}(K)} F_{K,\sigma}$

- discrete mass balance on the dual mesh

$$\boxed{\frac{|D_\sigma|}{\delta t}(\rho_{D_\sigma}^{n+1} - \rho_{D_\sigma}^n) + \sum_{\epsilon\in\widetilde{\mathscr{E}}(D_\sigma)} F_{\sigma,\epsilon}(\rho^{n+1}, u^{n+1}) = 0} \qquad \forall\sigma\in\mathscr{E}_{\mathrm{int}}.$$

Discrete energy estimate - III

- local-in-time energy inequality

$$\frac{1}{2}\sum_{\sigma\in\mathcal{E}_{\rm int}} |D_\sigma|\Big(\rho_{D_\sigma}^{n+1}\ |u_\sigma^{n+1}|^2 - \rho_{D_\sigma}^{n}\ |u_\sigma^{n}|^2\Big) + \sum_{K\in\mathcal{M}} |K|\Big(H_{\varepsilon,h}(\rho_K^{n+1}) - H_{\varepsilon,h}(\rho_K^{n})\Big)$$
$$+\mu\ \delta t\ \|u^{n+1}\|_{1,2,\mathcal{M}}^2 + \underbrace{\mathcal{R}^{n+1}}_{\geq 0} \leq 0$$

- given (ρ^n, u^n), existence of (ρ^{n+1}, u^{n+1}) by a fixed point argument
  - need to truncate the pressure $p_\varepsilon \rightsquigarrow p_{\varepsilon,h}$

- global discrete energy inequality

$$\frac{1}{2}\sum_{\sigma\in\mathcal{E}_{\rm int}} |D_\sigma|\ \rho_{D_\sigma}^{n}\ |u_\sigma^{n}|^2 + \sum_{K\in\mathcal{M}} |K|\, H_{\varepsilon,h}(\rho_K^{n}) + \mu \sum_{k=1}^{n} \delta t\ \|u^k\|_{1,2,\mathcal{M}}^2$$
$$\leq \frac{1}{2}\sum_{\sigma\in\mathcal{E}_{\rm int}} |D_\sigma|\ \rho_{D_\sigma}^{0}\ |u_\sigma^{0}|^2 + \sum_{K\in\mathcal{M}} |K|\, H_{\varepsilon,h}(\rho_K^{0})$$

Maximal constraint on the density

$$H'_{\varepsilon,h}(\rho)\rho - H_{\varepsilon,h}(\rho) = p_{\varepsilon,h}(\rho) \quad \text{with } p_{\varepsilon,h}(\rho) = \begin{cases} p_\varepsilon(\rho) & \text{if } \rho \le 1-\delta_{\varepsilon,h} \\ \varepsilon\delta_{\varepsilon,h}^{-\beta}\rho^\gamma & \text{if } \rho > 1-\delta_{\varepsilon,h} \end{cases}$$

- the control of the internal energy $\sum_{K\in\mathcal{M}} |K|\, H_{\varepsilon,h}(\rho_K^n) \le E_0$ implies

$$\text{meas}\,\{\rho^n > 1-\delta_{\varepsilon,h}\} \le C(E_0,\gamma,\beta)\varepsilon^{-1}\delta_{\varepsilon,h}^{\beta-1}$$

- choose $\delta_{\varepsilon,h} = \varepsilon^{\frac{1}{\beta-1}} h^\alpha$ with $\alpha(\beta-1) > d$

$$\Longrightarrow \text{meas}\,\{\rho^n > 1-\delta_{\varepsilon,h}\} \le C(E_0,\gamma,\beta)h^{\alpha(\beta-1)}$$

$$\Longrightarrow \text{meas}\,\{\rho^n > 1-\delta_{\varepsilon,h}\} < |K| \quad \forall\ K\in\mathcal{M},\ \text{for } h \text{ small enough}$$

conclusion

$$\boxed{\rho_K^n \le 1-\delta_{\varepsilon,h} = 1-h^\alpha\varepsilon^{\frac{1}{\beta-1}}} \quad \text{for all } K\in\mathcal{M}$$

Control of the pressure - I

- $\rho_K^n \leq 1 - \delta_{\varepsilon,h} = 1 - h^{\alpha}\varepsilon^{\frac{1}{\beta-1}}$ for all $K \in \mathcal{M}$ but $p_\varepsilon(1-\delta_{\varepsilon,h}) \propto \varepsilon^{-\frac{1}{\beta-1}}$
- continuous case: test the momentum eq. against $\psi = \mathcal{B}(\rho_\varepsilon - <\rho_\varepsilon>)$, $\mathcal{B} \approx \mathrm{div}^{-1}$

Lemma (L^q discrete inf-sup property)

Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a staggered discretization of Ω. Then, there exists a linear operator

$$\mathcal{B}_{\mathcal{M}} : L_{\mathcal{M},0}(\Omega) \longrightarrow H_{\mathcal{M},0}(\Omega)$$

depending only on Ω and on the discretization such that the following properties hold:

(i) *For all $f \in L_{\mathcal{M},0}(\Omega)$,*

$$\int_\Omega p \, \mathrm{div}_{\mathcal{M}}(\mathcal{B}_{\mathcal{M}} f) dx = \int_\Omega p f \, dx, \qquad \forall p \in L_{\mathcal{M}}(\Omega).$$

(ii) *For all $q \in (1, +\infty)$, there exists $C = C(q, d, \Omega)$, such that*

$$\|\mathcal{B}_{\mathcal{M}} f\|_{1,q,\mathcal{M}} \leq C \|f\|_{L^q(\Omega)}.$$

Control of the pressure - II

- we set

$$\psi^{n+1} = \mathscr{B}_{\mathcal{M}}(\rho^{n+1} - <\rho^{n+1}>) \quad \text{where} \quad <\rho^{n+1}> = \frac{1}{|\Omega|} \sum_{K \in \mathcal{M}} |K| \rho_K^{n+1} = \frac{1}{|\Omega|} \int_\Omega \rho_\varepsilon^0(x)\, dx$$

- thanks to the energy estimate, there exists $C > 0$ independent of ε such that

$$\left| \int_\Omega p_\varepsilon^{n+1}\ \mathrm{div}_{\mathcal{M}} \psi^{n+1}\ dx \right| = \left| \int_\Omega p_\varepsilon^{n+1} (\rho^{n+1} - <\rho^{n+1}>)\ dx \right| \leq C$$

- assuming that $<\rho_\varepsilon^0> \leq \rho^* < 1$ for some ρ^* independent of ε

$$\boxed{\| p_\varepsilon(\rho_\varepsilon^{n+1}) \|_{L^1(\Omega)} \leq C}$$

- all the discrete norms are equivalent, hence there exists $C_h > 0$ indep. of ε such that

$$p_K^n \leq C_h \ \ \forall\ K \in \mathcal{M} \quad \text{and} \quad \rho_K^n \leq 1 - C_h \varepsilon^{\frac{1}{\beta}} \ \ \forall\ K \in \mathcal{M}$$

Asymptotic behavior as $\varepsilon \to 0$

- thanks to the previous estimates
  - $(\rho_\varepsilon)_\varepsilon$ bounded in $L^\infty([0,T]\times\Omega)$
  - $(u_\varepsilon)_\varepsilon$ bounded in $L^2([0,T];(H_0^1(\Omega))^d)$
  - $(p_\varepsilon(\rho_\varepsilon))_\varepsilon$ bounded in $L^1([0,T]\times\Omega)$
- by the Bolzano-Weierstrass theorem (fixed mesh $\implies$ finite dimension), there exists a converging subsequence $(\rho_\varepsilon, u_\varepsilon, p_\varepsilon(\rho_\varepsilon))_\varepsilon$
- maximal density constraint satisfied by the limit ρ:

$$0 \le \rho_K^n \le 1 \qquad \forall\ K \in \mathcal{M},\ n = 0, \dots N$$

- exclusion relation:

$$(1-\rho_{\varepsilon,K}^n)p_{\varepsilon,K}^n = \varepsilon^{1/\beta}(\rho_{\varepsilon,K}^n)^{\gamma/\beta}(p_{\varepsilon,K}^n)^{\frac{\beta-1}{\beta}} \underset{\varepsilon\to 0}{\to} 0$$

$$\implies (1-\rho_K^n)p_K^n = 0 \qquad \forall\ K \in \mathcal{M},\ n = 1, \dots N$$

Pressure correction scheme

Knowing $(\rho^{n-1}, \rho^n, u^n) \in L_{\mathcal{M}}(\Omega) \times L_{\mathcal{M}}(\Omega) \times H_{\mathcal{M},0}(\Omega)$, compute (ρ^{n+1}, u^{n+1}) as follows:

Pressure gradient scaling step:

$$(\overline{\nabla p_\varepsilon})^n_\sigma = \left(\frac{\rho^n_{D_\sigma}}{\rho^{n-1}_{D_\sigma}} \right)^{1/2} (\nabla p_\varepsilon)^n_\sigma \qquad \forall \sigma \in \mathcal{E}_{\mathrm{int}}$$

Prediction step: Solve for $\widetilde{u}^{n+1} \in H_{\mathcal{M},0}(\Omega)$: $\forall \sigma \in \mathcal{E}_{\mathrm{int}}$

$$\frac{1}{\delta t}\left(\rho^n_{D_\sigma} \widetilde{u}^{n+1}_\sigma - \rho^{n-1}_{D_\sigma} u^n_\sigma\right) + \frac{1}{|D_\sigma|} \sum_{\epsilon \in \widetilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}(\rho^n, u^n)\ \widetilde{u}^{n+1}_\epsilon + (\overline{\nabla p_\varepsilon})^n_\sigma - \mathrm{div}\ \tau(\widetilde{u}^{n+1})_\sigma = 0,$$

Correction step: Solve for $\rho^{n+1} \in L_{\mathcal{M}}(\Omega)$ and $u^{n+1} \in H_{\mathcal{M},0}(\Omega)$:

$$\frac{1}{\delta t}\left(\rho^{n+1}_K - \rho^n_K\right) + \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}(\rho^{n+1}, u^{n+1}) = 0 \qquad \forall K \in \mathcal{M},$$

$$\frac{1}{\delta t}\rho^n_{D_\sigma}\left(u^{n+1}_\sigma - \widetilde{u}^{n+1}_\sigma\right) + (\nabla p_\varepsilon)^{n+1}_\sigma - (\overline{\nabla p_\varepsilon})^n_\sigma = 0 \qquad \forall \sigma \in \mathcal{E}_{\mathrm{int}}.$$

Pressure correction scheme

- initialization of the scheme

$$\rho_K^{-1} = \frac{1}{|K|}\int_K \rho_0^\varepsilon(x)\ dx \quad \forall K \in \mathcal{M}, \qquad u_\sigma^0 = \frac{1}{|\sigma|}\int_\sigma u_0^\varepsilon(x) d\sigma(x) \quad \forall \sigma \in \mathcal{E}_{\rm int}$$

we compute ρ^0 by solving the mass balance equation for $n=-1$:

$$\frac{1}{\delta t}\left(\rho_K^0 - \rho_K^{-1}\right) + \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}(K)} F_{K,\sigma}(\rho^0, u^0) = 0 \qquad \forall K \in \mathcal{M}.$$

- we assume that $(\rho_\varepsilon^0, u_\varepsilon^0)$ is "well-prepared":

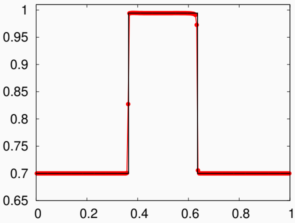
$$\|u_0^\varepsilon\|_{H^1(\Omega)^d} + \|p_\varepsilon(\rho_0^\varepsilon)\|_{L^1(\Omega)} + \|(1-\rho_0^\varepsilon)^{-1}[\mathrm{div}\ u_0^\varepsilon]_-\|_{L^2(\Omega)} \le C$$

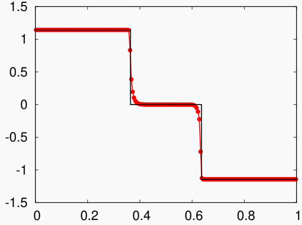
$\Longrightarrow$ appropriate bounds on ρ_K^0 + control of the discrete energy

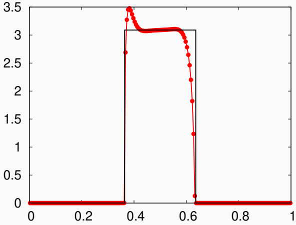
Numerical simulations with CALIF^{3}S - inviscid 1D case

- Riemann problem ($\varepsilon > 0$ fixed): $(\rho_\varepsilon^0, (\rho u)_\varepsilon^0) = \begin{cases} (0.7, 0.8) & x \in [0, 0.5[\\ (0.7, -0.8) & x \in]0.5, 1] \end{cases}$

$$\gamma = 2,\ \beta = 2, \varepsilon = 10^{-4},\ \Delta x = 5.10^{-3},\ \Delta t = 5.10^{-4}$$

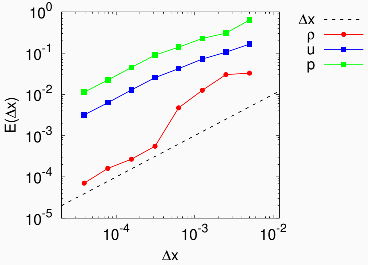






Mesh refinement

- error in L^1-norm

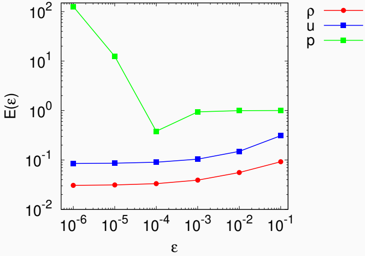


meshes composed by 200×2^n cells, $n = 0, 1, \ldots 7$

the time step is adapted so that the CFL keeps the same value (1.73)

Limit $\varepsilon \to 0$ - comparison with the limit exact solution

- error in L^1-norm

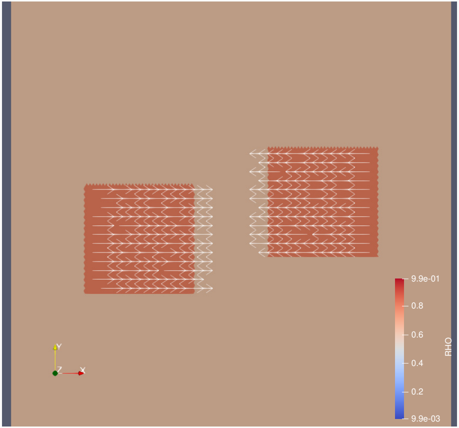


ε	CPU-time (s)	$E_\rho(\varepsilon)$	$E_u(\varepsilon)$	$E_p(\varepsilon)$
10^{-1}	0.7724	0.092	0.31	1.0
10^{-2}	0.908	0.056	0.15	0.99
10^{-3}	0.9753	0.039	0.10	0.94
10^{-4}	1.059	0.033	0.090	0.38
10^{-5}	1.109	0.031	0.086	12
10^{-6}	1.245	0.031	0.084	127

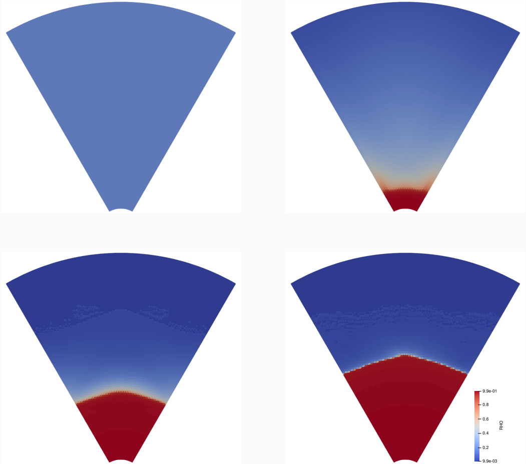
Numerical simulations with CALIF^{3}S - collision in 2D

$$\rho^0 = 0.8\ 1_{\Omega_1} + 0.6\ 1_{\Omega_2}$$

$$\varepsilon = 10^{-4},\ h = 5.10^{-3},\ \Delta t = 5.10^{-4}$$



2D Convergent corridor - add $f = \nabla D - u$



12020 triangles, $\delta t = 10^{-2}$, $\varepsilon = 0.04$

CPU time: 23 min for 1000 time iterations

Conclusion

- FV-FE staggered scheme compatible with the limit $\varepsilon \to 0$
  - control of the pressure thanks to the discrete inf-sup condition
- numerical simulations performed with CALIF^3S developed by IRSN
- convergence as $h \to 0$ (ε fixed): open pb for the compressible Navier-Stokes eq.
  - for the stationary case: see P., Saleh '20

THANK YOU !

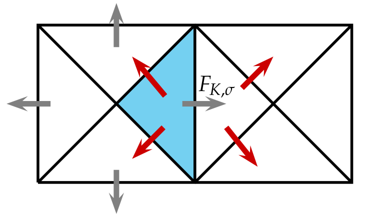
Discrete operators - velocity convection operator

$$\operatorname{div}_{\mathcal{E}}(\rho u \otimes u)(x) = \sum_{\sigma \in \mathcal{E}_{\text{int}}} \frac{1}{|D_\sigma|} \Big(\sum_{\epsilon \in \widetilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}(\rho, u) u_\epsilon \Big) \chi_{D_\sigma}(x)$$

$F_{\sigma,\epsilon}$ is defined so that the following discrete mass eq is satisfied on the dual mesh

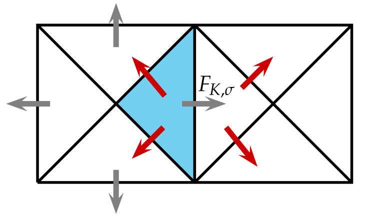
$$\frac{|D_\sigma|}{\delta t} (\rho_{D_\sigma}^{n+1} - \rho_{D_\sigma}^{n}) + \sum_{\epsilon \in \widetilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}(\rho^{n+1}, u^{n+1}) = 0 \quad \forall \sigma \in \mathcal{E}_{\text{int}}$$

where $|D_\sigma| \rho_{D_\sigma} = |D_{K,\sigma}| \rho_{D_{K,\sigma}} + |D_{L,\sigma}| \rho_{D_{L,\sigma}}$



mass conservation on the primal mesh

$$\sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma} + \frac{|K|}{\delta t} (\rho_K^{n+1} - \rho_K^{n}) = 0 \quad \forall K \in \mathcal{M}$$



- let w be such that $\operatorname{div} w = \mathrm{cst}$, $\int_\sigma w \cdot n_{K,\sigma} = F_{K,\sigma}$ and define $F_{\sigma,\epsilon} = \int_\epsilon w \cdot n_{\sigma,\epsilon}$
- then, using the mass conservation on the cell K:

$$F_{K,\sigma} + \sum_{\substack{\epsilon \in \widetilde{\mathscr{E}}(D_\sigma) \\ \epsilon \subset K}} F_{\sigma,\epsilon} = \int_{D_{K,\sigma}} \operatorname{div} w = \frac{|D_{K,\sigma}|}{|K|} \int_K \operatorname{div} w = \frac{|D_{K,\sigma}|}{|K|} \sum_{\sigma' \in \mathscr{E}(K)} F_{K,\sigma'}$$

$$= -\frac{|D_{K,\sigma}|}{\delta t} (\rho_K^{n+1} - \rho_K^n)$$

- idem for the cell L and we sum the two equations

$$\Longrightarrow \quad \underbrace{F_{K,\sigma} + F_{L,\sigma}}_{=0} + \sum_{\epsilon \in \widetilde{\mathscr{E}}(D_\sigma)} F_{\sigma,\epsilon}$$

$$= -\frac{1}{\delta t} \Big(\underbrace{\big(|D_{K,\sigma}|\rho_K^{n+1} + |D_{L,\sigma}|\rho_L^{n+1}\big)}_{=|D_\sigma|\rho_{D_\sigma}^{n+1}} - |D_{K,\sigma}|\rho_K^n - |D_{L,\sigma}|\rho_L^n \Big)$$

Degond, Hua, Navoret: IMEX colocated FV scheme

$$\begin{cases} \dfrac{\rho^{n+1}-\rho^n}{\Delta t} + \mathrm{div}(\rho u)^{n+1} = 0 \\ \dfrac{(\rho u)^{n+1}-(\rho u)^n}{\Delta t} + \mathrm{div}(\rho u \otimes u)^n + \nabla p_\varepsilon^i(\rho^{n+1}) + \nabla p_\varepsilon^e(\rho^n) = 0 \end{cases}$$

- <u>reformulation</u>: div(Mom eq) and insert the result into Mass eq.

$$\begin{cases} \dfrac{\rho^{n+1}-\rho^n}{\Delta t} + \mathrm{div}(\rho u)^n - \Delta t \Delta p_\varepsilon^i(\rho^{n+1}) - \Delta t \nabla^2 : (\rho u \otimes u)^n - \Delta t \Delta p_\varepsilon^e(\rho^n) = 0 \\ \dfrac{(\rho u)^{n+1}-(\rho u)^n}{\Delta t} + \mathrm{div}(\rho u \otimes u)^n + \nabla p_\varepsilon^i(\rho^{n+1}) + \nabla p_\varepsilon^e(\rho^n) = 0 \end{cases}$$

$\rightarrow$ uniform stability condition

- $\rho^{n+1} = \rho(p_\varepsilon^{n+1})$ $\rightarrow$ nonlinear elliptic equation on $(p_\varepsilon^i)^{n+1}$

1) compute $(p_\varepsilon^i)^{n+1}$
2) deduce the new density ρ^{n+1} $\rightarrow$ ρ^{n+1} satisfies automatically the constraint
3) compute the new momentum $(\rho u)^{n+1}$

Computation of the momentum, Gauge Method

- Gauge Decomposition $\quad \rho u = a - \nabla\varphi, \quad \mathrm{div}\, a = 0$

- time discretization

$$\Delta\varphi^{n+1} = \frac{1}{\Delta t}(\rho^{n+1} - \rho^n) \qquad {\varphi^{n+1}}_{|\partial\Omega} = 0$$

$$\Delta P^{n+1} = -\nabla^2 : (\rho u \otimes u)^n \qquad \rightsquigarrow \quad P^{n+1} = p_\varepsilon(\rho^{n+1}) - \frac{\varphi^{n+1} - \varphi^n}{\Delta t}$$

$$\frac{a^{n+1} - a^n}{\Delta t} + \mathrm{div}(\rho u \otimes u)^n + \nabla P^{n+1} = 0$$

$$(\rho u)^{n+1} = a^{n+1} - \nabla\varphi^{n+1}$$

Time-Space discretization in 1D

$$\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} + \frac{1}{\Delta x}\left[Q_{j+1/2}^{n+1/2} - Q_{j-1/2}^{n+1/2} \right] = 0$$

$$\frac{(\rho u)_j^{n+1} - (\rho u)_j^n}{\Delta t} + \frac{1}{\Delta x}\left[F_{j+1/2}^n - F_{j-1/2}^n\right] + \frac{1}{2\Delta x}\left[p_\varepsilon(\rho_{j+1}^{n+1}) - p_\varepsilon(\rho_{j-1}^{n+1}) \right] = 0$$

with

$$Q_{j+1/2}^{n+1/2} = \frac{1}{2}\left[(\rho u)_j^{n+1} + (\rho u)_{j+1}^{n+1}\right] - \frac{D_{j+1/2}^n}{2}(\rho_{j+1}^n - \rho_j^n)$$

$$F_{j+1/2}^n = \frac{1}{2}\left[\frac{\left((\rho u)_j^n\right)^2}{\rho_j^n} + \frac{\left((\rho u)_{j+1}^n\right)^2}{\rho_{j+1}^n} \right] - \frac{D_{j+1/2}^n}{2}\left((\rho u)_{j+1}^n - (\rho u)_j^n\right)$$

$$D_{j+1/2}^n = \max\left\{ |u_j^n|, |u_{j+1}^n| \right\}$$

$$\rho\big((p_\varepsilon)_j^{n+1}\big) - \frac{\Delta t^2}{4\Delta x^2}\Big[p_\varepsilon(\rho_{j+2}^{n+1}) - 2p_\varepsilon(\rho_j^{n+1}) + p_\varepsilon(\rho_{j-2}^{n+1})\Big]$$

$$= \rho_j^n - \frac{\Delta t}{2\Delta x}\big((\rho u)_{j+1}^n - (\rho u)_{j-1}^n\big) + \frac{\Delta t^2}{2\Delta x^2}\Big[F_{j+3/2}^n - F_{j+1/2}^n - F_{j-1/2}^n + F_{j-3/2}^n\Big]$$

$$+ \frac{\Delta t}{2\Delta x}\Big[D_{j+1/2}^n(\rho_{j+1}^n - \rho_j^n) - D_{j-1/2}^n(\rho_j^n - \rho_{j-1}^n)\Big]$$

$$\frac{1}{\Delta x^2}\Big[\varphi_{j+1}^{n+1} - 2\varphi_j^{n+1} + \varphi_{j-1}^{n+1}\Big]$$

$$= \frac{1}{\Delta t}(\rho_j^{n+1} - \rho_j^n) - \frac{1}{2\Delta x}\Big[D_{j+1/2}^n(\rho_{j+1}^n - \rho_j^n) - D_{j-1/2}^n(\rho_j^n - \rho_{j-1}^n)\Big]$$

$$a^{n+1} = a^n - \Delta t\Big((\rho u \otimes u)^n + p_\varepsilon(\rho^{n+1})\Big)\Big|_0^1$$

$$+ \frac{\Delta t}{2}\sum_{j=1}^{Nx}\Big[D_{j+1/2}^n(\rho_{j+1}^n - \rho_j^n) - D_{j-1/2}^n(\rho_j^n - \rho_{j-1}^n)\Big]$$

$$(\rho u)_j^{n+1} = a^{n+1} - \frac{1}{2\Delta x}\Big[\varphi_{j+1}^{n+1} - \varphi_{j-1}^{n+1}\Big]$$